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Cesàro Summability of Two-Parameter Trigonometric-Fourier Series*

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The two-dimensional classical Hardy spaces $H_p(\mathbf{T}\times\mathbf{T})$ on the bidisc are introduced and it is shown that the maximal operator of the Cesàro means of a distribution is bounded from $H_p(\mathbf{T}\times\mathbf{T})$ to $L_p(\mathbf{T}^2)$ $(3/4<p\leqslant\infty)$ and is of weak type $(H_1^\sharp(\mathbf{T}\times\mathbf{T}), L_1(\mathbf{T}^2))$ where the Hardy space $H_1^\sharp(\mathbf{T}\times\mathbf{T})$ is defined by the hybrid maximal function. As a consequence we obtain that the Cesàro means of a function $f\in H_1^\sharp(\mathbf{T}\times\mathbf{T})\supset L\log L(\mathbf{T}^2)$ converge a.e. to the function in question. © 1997 Academic Press

1. INTRODUCTION

For double trigonometric Fourier series Marcinkievicz and Zygmund [14] proved that the Cesàro means $\sigma_{n,m}f$ of a function $f\in L_1(\mathbf{T}^2)$ converge a.e. to f as $n, m\to\infty$, provided that the pairs (n, m) are in a positive cone, i.e., provided that $2^{-\delta}\leqslant n/m\leqslant 2^{\delta}$ for any $\delta\geqslant 0$. A new proof of this result was given by the author [19]. Moreover, Zygmund [24] verified that if $f\in L\log L(\mathbf{T}^2)$ then the two-parameter Cesàro summability holds.

We proved in [20] and [19] that, in the one-dimensional case, the maximal operator of the Cesàro means of a distribution is bounded from the Hardy–Lorentz space $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ if $3/4<p\leqslant\infty$, $0<q\leqslant\infty$, and that, in the two-dimensional case, it is bounded from $H_{p,q}(\mathbf{T}^2)$ $(\neq H_{p,q}(\mathbf{T}\times\mathbf{T}))$ to $L_{p,q}(\mathbf{T}^2)$ if $5/6<p\leqslant\infty$, $0<q\leqslant\infty$, provided that the supremum in the maximal operator is taken over a positive cone.

In this paper we generalize these results for the unrestricted maximal operator of the two-parameter trigonometric Fourier series. The analogous result for a two-parameter Walsh–Fourier series has been shown by the

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author [21]. The Hardy–Lorentz spaces $H_{p,q}(\mathbf{T}\times\mathbf{T})$ of distributions are introduced with the $L_{p,q}(\mathbf{T}^2)$ Lorentz norms of the two-dimensional non-tangential maximal function. Of course, $H_p(\mathbf{T}\times\mathbf{T})=H_{p,p}(\mathbf{T}\times\mathbf{T})$ are the usual Hardy spaces $(0<p\leqslant\infty)$. We will show that the maximal operator of the Cesàro means of a distribution is bounded from $H_{p,q}(\mathbf{T}\times\mathbf{T})$ to $L_{p,q}(\mathbf{T}^2)$ $(3/4<p\leqslant\infty,\ 0<q\leqslant\infty)$ and is of weak type $(H_1^\sharp(\mathbf{T}\times\mathbf{T}), L_1(\mathbf{T}^2))$, i.e.,

$$\sup_{\gamma>0}\gamma\lambda(\sup_{n,m\in\mathbf{N}}|\sigma_{n,m}f|>\gamma)\leqslant C\,\|f\|_{H_1^\sharp(\mathbf{T}\times\mathbf{T})}\qquad (f\in H_1^\sharp(\mathbf{T}\times\mathbf{T})).$$

A usual density argument implies then that the Cesàro means $\sigma_{n,m}f$ converge a.e. to f as $n,\ m\to\infty$ whenever $f\in H_1^\sharp(\mathbf{T}\times\mathbf{T})\supset L\log L(\mathbf{T}^2)$. This last result can be found also in Weisz [20].

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2. PRELIMINARIES AND NOTATIONS

For a set $\mathbf{X}\neq\emptyset$ let $\mathbf{X}^2$ be its Cartesian product $\mathbf{X}\times\mathbf{X}$ taken with itself, moreover, let $\mathbf{T}:=[-\pi,\pi)$ and λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write L_p instead of the real $L_p(\mathbf{T}^2,\lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p:=(\int_{\mathbf{T}^2}|f|^p\,d\lambda)^{1/p}$ $(0<p\leqslant\infty)$.

The distribution function of a Lebesgue-measurable function f is defined by

$$\lambda(\{|f|>\gamma\}):=\lambda(\{x:|f(x)|>\gamma\})\qquad(\gamma\geqslant 0).$$

The *weak* L_p space $L_p^*(0<p<\infty)$ consists of all measureable functions f for which

$$\|f\|_{L_p^*}:=\sup_{\gamma>0}\gamma\lambda(\{|f|>\gamma\})^{1/p}<\infty$$

while we set $L_\infty^*=L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\tilde{f}(t):=\inf\{\gamma:\lambda(\{|f|>\gamma\})\leqslant t\}.$$

Lorentz space $L_{p,q}$ is defined as follows: for $0<p<\infty$, $0<q<\infty$

$$\|f\|_{p,q}:=\left(\int_0^\infty\tilde{f}(t)^q\,t^{q/p}\,\frac{dt}{t}\right)^{1/q}$$

while for $0<p\leqslant\infty$

$$\|f\|_{p,\infty}:=\sup_{t>0} t^{1/p}\tilde{f}(t).$$

Let

$$L_{p,q}:=L_{p,q}(\mathbf{T}^j,\lambda):=\{f: \|f\|_{p,q}<\infty\} \qquad (j=1,2).$$

One can show the following equalities:

$$L_{p,p}=L_p, \qquad L_{p,\infty}=L_p^* \qquad (0<p\leqslant\infty)$$

(see e.g. Bennett, Sharpley [1] or Bergh, Löfström [2]).

Let f be a distribution on $C^\infty(\mathbf{T}^2)$ (briefly $f\in\mathscr{D}'(\mathbf{T}^2)=\mathscr{D}'$). The (n,m)th Fourier coefficient is defined by $\hat{f}(n,m):=f(e^{-\imath nx}e^{-\imath my})$ where $\imath=\sqrt{-1}$. In special case, if f is an integrable function then

$$\hat{f}(n,m)=\frac{1}{(2\pi)^2}\int_{\mathbf{T}}\int_{\mathbf{T}} f(x,y)\, e^{-\imath nx}e^{-\imath my}\,dx\,dy.$$

For simplicity, we assume that, for a distribution $f\in\mathscr{D}'$, we have $\hat{f}(n,0)=\hat{f}(0,n)=0$ $(n\in\mathbf{N})$. Denote by $s_{n,m}f$ the (n,m)th partial sum of the Fourier series of a distribution f, namely,

$$s_{n,m}f(x):=\sum_{k=-n}^{n}\sum_{l=-m}^{m}\hat{f}(k,l)\,e^{\imath kx}e^{\imath ly}.$$

For $f\in\mathscr{D}'$ and $z_1:=re^{\imath x}$, $z_2:=se^{\imath y}$ $(0<r,\ s<1)$ let

$$u(z_1,z_2)=u(re^{\imath x},se^{\imath y}):=(f*P_r\times P_s)(x,y)$$

where $*$ denotes the convolution and

$$P_r(x):=\sum_{k=-\infty}^{\infty} r^{|k|}e^{\imath kx}=\frac{1-r^2}{1+r^2-2r\cos x} \qquad (x\in\mathbf{T})$$

is the Poisson kernel. It is easy to show that $u(z_1,z_2)$ is a biharmonic function on the bidisc and

$$u(re^{\imath x},se^{\imath y})=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\hat{f}(k,l)\,r^{|k|}s^{|l|}e^{\imath kx}e^{\imath ly}$$

with absolute and uniform convergence (see e.g. Gundy, Stein [12], Edwards [8]).

Let $0<\alpha<1$ be an arbitrary number. We denote by $\Omega_\alpha(x)$ $(x\in\mathbf{T})$ the region bounded by two tangents to the circle $|z|=\alpha$ from $e^{\imath x}$ and the longer arc of the circle included between the points of tangency. The non-tangential maximal function is defined by

$$u^*_{\alpha,\beta}(x,y):=\sup_{z_1\in\Omega_\alpha(x)}\sup_{z_2\in\Omega_\beta(y)}|u(z_1,z_2)| \qquad (0<\alpha,\beta<1).$$

For $0<p,q\leqslant\infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbf{T}\times\mathbf{T})=H_{p,q}$ consists of all distributions f for which $u^*_{\alpha,\beta}\in L_{p,q}$ and set

$$\|f\|_{H_{p,q}}:=\|u^*_{1/2,1/2}\|_{p,q}.$$

It is known that if $f\in H_p$ $(0<p<\infty)$ then $f(x,y)=\lim_{r,s\to 1}u(re^{\imath x},se^{\imath y})$ in the sense of distributions (see Gundy, Stein [12]).

Let us introduce the hybrid Hardy spaces. For $f\in L_1(\mathbf{T}^2)$ and $z:=re^{\imath x}$ $(0<r<1)$ let

$$v(z,y)=v(re^{\imath x},y):=\frac{1}{2\pi}\int_{\mathbf{T}}f(t,y)\,P_r(x-t)\,dt$$

and

$$v^+_\alpha(x,y):=\sup_{z\in\Omega_\alpha(x)}|v(z,y)| \qquad (0<\alpha<1).$$

We say that $f\in L_1(\mathbf{T}^2)$ is in the hybrid Hardy–Lorentz space $H^\sharp_{p,q}(\mathbf{T}\times\mathbf{T})=H^\sharp_{p,q}$ if

$$\|f\|_{H^\sharp_{p,q}}:=\|v^+_{1/2}\|_{p,q}<\infty.$$

The equivalences $\|u^*_{\alpha,\beta}\|_{p,q}\sim\|u^*_{1/2,1/2}\|_{p,q}$, $\|v^+_\alpha\|_{p,q}\sim\|v^+_{1/2}\|_{p,q}$ $(0<p,q<\infty,$ $0<\alpha,\beta<1)$ and $H_{p,q}\sim H^\sharp_{p,q}\sim L_{p,q}$ $(1<p<\infty,\ 0<q\leqslant\infty)$ were proved in Fefferman, Stein [9], Gundy, Stein [12] and Lin [13]. Note that in case $p=q$ the usual definition of Hardy spaces $H_{p,p}=H_p$ and $H^\sharp_{p,p}=H^\sharp_p$ are obtained. For other equivalent definitions we call for Gundy, Stein [12], Gundy [11] and Chang, Fefferman [4].

In this paper the constants C are absolute constants and the constants C_p (resp. $C_{p,q}$) are depending only on p (resp. p and q) and may denote different constants in different contexts.

Recall that, in the one-dimensional case, $L_1(\mathbf{T})\subset H_{1,\infty}(\mathbf{T})$ and $L\log L(\mathbf{T})\subset H_1(\mathbf{T})$, more exactly,

$$\|f\|_{H_{1,\infty}(\mathbf{T})}=\sup_{\gamma>0}\gamma\lambda(u^*_{1/2}>\gamma)\leqslant C\,\|f\|_1 \qquad (f\in L_1(\mathbf{T})) \tag{1}$$

and

$$\|f\|_{H_1(\mathbf{T})} \leqslant C + C \, \| |f| \log^+ |f| \|_1 \qquad (f \in L \log L(\mathbf{T})) \tag{2}$$

where $\log^+ u = 1_{\{u>1\}} \log u$ (see Fefferman, Stein [9] and Stein [18]).
These results are generalized for two parameters in the following way.

THEOREM 1. *We have* $L \log L \subset H_1^\sharp \subset H_{1,\infty}$ *more exactly,*

$$\|f\|_{H_{1,\infty}} = \sup_{\gamma>0} \gamma\lambda(u^*_{1/2,\,1/2} > \gamma) \leqslant C \, \|f\|_{H_1^\sharp} \qquad (f \in H_1^\sharp) \tag{3}$$

and

$$\|f\|_{H_1^\sharp} \leqslant C + C \, \| |f| \log^+ |f| \|_1 \qquad (f \in L \log L). \tag{4}$$

Proof. Applying Fubini's theorem, (1) and the positivity of the Poisson kernel we have

$$\lambda\left((x,y) : \sup_{re^{\imath v} \in \Omega_{1/2}(x)} \sup_{se^{\imath w} \in \Omega_{1/2}(y)} \left| \int_{\mathbf{T}} \int_{\mathbf{T}} f(t,u)\, P_r(v-t)\, P_s(w-u)\, dt\, du \right| > \gamma \right)$$

$$\leqslant \lambda\left((x,y) : \sup_{se^{\imath w} \in \Omega_{1/2}(y)} \int_{\mathbf{T}} \left(\sup_{re^{\imath v} \in \Omega_{1/2}(x)} \left| \int_{\mathbf{T}} f(t,u)\, P_r(v-t)\, dt \right| \right) \right.$$

$$\left. \times P_s(w-u)\, du > \gamma \right)$$

$$= \int_{\mathbf{T}} \int_{\mathbf{T}} 1_{\{\sup_{se^{\imath w} \in \Omega_{1/2}(\cdot)} \int_{\mathbf{T}} (\sup_{re^{\imath v} \in \Omega_{1/2}(\cdot)} |\int_{\mathbf{T}} f(t,u)\, P_r(v-t)\, dt|)\, P_s(w-u) du > \gamma\}}(x,y)\, dy\, dx$$

$$\leqslant \frac{C}{\gamma} \int_{\mathbf{T}} \int_{\mathbf{T}} \sup_{re^{\imath v} \in \Omega_{1/2}(x)} \left| \int_{\mathbf{T}} f(t,y)\, P_r(v-t)\, dt \right| dy\, dx$$

$$= \frac{C}{\gamma} \int_{\mathbf{T}} \int_{\mathbf{T}} v^+_{1/2}(x,y)\, dx\, dy$$

which proves (3). (4) comes easily from (2). ∎

3. ATOMIC DECOMPOSITION AND BOUNDED OPERATORS ON HARDY SPACES

A *generalized interval* on $\mathbf{T}$ is either an interval $I \subset \mathbf{T}$ or $I = [-\pi, x) \cup [y, \pi)$. A *generalized rectangle* on $\mathbf{T}^2$ is the Descartes product $I \times J$ of two generalized intervals.

A function $a \in L_2$ is a *p-atom* if

(i) supp $a \subset F$ for an open set $F \subset \mathbf{T}^2$

(ii) $$\|a\|_2 \leqslant \lambda(F)^{1/2-1/p}$$

(iii) $a = \sum_R \lambda_R a_R$ where the λ_R's are real numbers and the a_R's are functions (called "elementary particles") satisfying

(α) supp $a_R \subset R$ for any generalized rectangle $R = I \times J \subset F$

(β)

$$\left\| \frac{\partial^N a_R(x, y)}{\partial x^N} \right\|_\infty \leqslant \frac{C}{\sqrt{|R|}\,|I|^N} \qquad \text{and} \qquad \left\| \frac{\partial^N a_R(x, y)}{\partial y^N} \right\|_\infty \leqslant \frac{C}{\sqrt{|R|}\,|J|^N}$$

for all $N \leqslant [2/p - 1/2]$

(γ) for all $x, y \in \mathbf{T}$ and all $M \leqslant [2/p - 3/2]$

$$\int_{\mathbf{T}} a_R(x, y)\, x^M \, dx = \int_{\mathbf{T}} a_R(x, y)\, y^M \, dy = 0$$

(δ) $$\left(\sum_R \lambda_R^2 \right)^{1/2} \leqslant \lambda(F)^{1/2-1/p}.$$

If $a \in L_2$ satisfies (i) with a generalized rectangle F, (ii) and (γ) then a is called a *rectangle p-atom.*

The basic result of the atomic decomposition is stated as follows (see Chang, Fefferman [4], Fefferman [10], Wilson [23] and also Weisz [21]).

THEOREM A. *A distribution f is in H_p $(0 < p \leqslant 1)$ if and only if there exist a sequence $(a_k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a_k = f \qquad \text{in the sense of distributions,}$$
$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{5}$$

Moreover, the following equivalence of norms holds:

$$\|f\|_{H_p} \sim \inf\left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p} \tag{6}$$

where the infimum is taken over all decompositions of f *of the form* (5).

If I is a generalized interval then let $2^r I$ be the generalized interval with the same center as I and with length $2^r\,|I|$ $(r \in \mathbf{N})$. For a generalized rectangle $R = I \times J$ let $2^r R = 2^r I \times 2^r J$.

Using Theorem A, (3) and the interpolation results given in Lin [13] we can prove the following theorem similarly to the Theorem of Fefferman [10] (see also Corollary 1 in Weisz [21]).

THEOREM 2. *Suppose that the operator* T *is sublinear and* $p_0 < p \leqslant 1$. *Furthermore, assume that there exists* $\delta > 0$ *such that for every rectangle* p*-atom* a *supported on the generalized rectangle* R *and for every* $r \geqslant 2$ *one has*

$$\int_{\mathbf{T}^2 \setminus 2^r R} |Ta|^p \, d\lambda \leqslant C_p 2^{-\delta r} \tag{7}$$

where C_p *is a constant depending only on* p. *If* T *is bounded from* L_p *to* L_p $(p = 2, \infty)$ *then*

$$\|Tf\|_{p,q} \leqslant C_{p,q}\, \|f\|_{H,q} \qquad (f \in H_{p,q})$$

for every $p_0 < p < \infty$ *and* $0 < q \leqslant \infty$. *Specially,* T *is of weak type* $(H_1^\sharp, L_1)$, *i.e. if* $f \in H_1^\sharp$ *then*

$$\|Tf\|_{1,\infty} = \sup_{\gamma \geqslant 0} \gamma \lambda(|Tf| > \gamma) \leqslant C\, \|f\|_{H_{1,\infty}} \leqslant C\, \|f\|_{H_1^\sharp}.$$

4. CESÀRO SUMMABILITY OF TWO-PARAMETER TRIGONOMETRIC-FOURIER SERIES

For $n, m \in \mathbf{N}$ and a distribution f the *Cesàro mean* of order (n, m) of the Fourier series of f is given by

$$\sigma_{n,m} f := \frac{1}{n+1} \frac{1}{m+1} \sum_{k=0}^{n} \sum_{l=0}^{m} s_{k,l} f = f * (K_n \times K_m) \qquad (n, m \in \mathbf{N})$$

where K_n is the Fejér kernel of order n. It is shown in Zygmund [24] that

$$0 \leqslant K_n(t) \leqslant \frac{\pi^2}{(n+1)\, t^2} \qquad (0 < |t| < \pi) \tag{8}$$

and

$$\int_{\mathbf{T}} K_n(t)\, dt = \pi. \tag{9}$$

For a distribution f we consider the maximal operator of the Cesàro means

$$\sigma^* f := \sup_{n,\, m \in \mathbf{N}} |\sigma_{n,\, m} f|$$

and prove our following main result.

THEOREM 3. *There are absolute constants C and $C_{p,\, q}$ such that*

$$\|\sigma^* f\|_{p,\, q} \leqslant C_{p,\, q} \, \|f\|_{H_{p,q}} \qquad (f \in H_{p,\, q}) \tag{10}$$

for every $3/4 < p < \infty$ and $0 < q \leqslant \infty$. Especially, if $f \in H_1^\sharp$ then

$$\lambda(\sigma^* f > \gamma) \leqslant \frac{C}{\gamma} \, \|f\|_{H_1^\sharp} \qquad (\gamma > 0). \tag{11}$$

Proof. It is proved in Zygmund [24] that

$$\|\sigma^* f\|_p \leqslant C_p \, \|f\|_p \qquad (1 < p \leqslant \infty). \tag{12}$$

So, by Theorem 2, the proof of Theorem 3 will be complete if we show that the operator σ^* satisfies (7) for each $3/4 < p \leqslant 1$.

Let a be an arbitrary rectangle p-atom with support $R = I \times J$ and

$$2^{-K-1} < |I|/\pi \leqslant 2^{-K}, \qquad 2^{-L-1} < |J|/\pi \leqslant 2^{-L} \qquad (K, L \in \mathbf{N}).$$

We can suppose that the center of R is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}]$$

and

$$[-\pi 2^{-L-2}, \pi 2^{-L-2}] \subset J \subset [-\pi 2^{-L-1}, \pi 2^{-L-1}].$$

To prove (7) for the operator σ^* we have to integrate $|\sigma^* a|^p$ over

$$\mathbf{T}^2\backslash 2^r R = (\mathbf{T}\backslash 2^r I)\times J \cup (\mathbf{T}\backslash 2^r I)\times(\mathbf{T}\backslash J)$$
$$\cup\, I\times(\mathbf{T}\backslash 2^r J)\cup(\mathbf{T}\backslash I)(\mathbf{T}\backslash 2^r J)$$

where $r \geqslant 2$ is an arbitrary integer. We do this in four steps.

Step 1: *Integrating over* $(\mathbf{T}\backslash 2^r I)\times J$. Obviously,

$$\begin{aligned}
&\int_{\mathbf{T}\backslash 2^r I}\int_J |\sigma^* a(x, y)|^p\,dx\,dy \\
&\quad\leqslant \sum_{|i|=2^{r-2}}^{2^K-1}\int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}}\int_J |\sigma^* a(x, y)|^p\,dx\,dy \\
&\quad\leqslant \sum_{|i|=2^{r-2}}^{2^K-1}\int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}}\int_J \sup_{n\geqslant r_i,\, m\in\mathbf{N}} |\sigma_{n,m} a(x, y)|^p\,dx\,dy \\
&\qquad+\sum_{|i|=2^{r-2}}^{2^K-1}\int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}}\int_J \sup_{n< r_i,\, m\in\mathbf{N}} |\sigma_{n,m} a(x, y)|^p\,dx\,dy \\
&\quad= (A)+(B)
\end{aligned}\tag{13}$$

where $r_i := [2^K/i^\alpha]$ $(i\in\mathbf{N})$ with $\alpha>0$ chosen later.

It is easy to see that

$$\frac{1}{(x-t)^2}\leqslant\frac{1}{(\pi i 2^{-K}-\pi 2^{-K-1})^2}\leqslant\frac{4}{\pi^2}\,\frac{2^{2K}}{i^2}\tag{14}$$

if $x\in[\pi i 2^{-K}, \pi(i+1)2^{-K})$ $(|i|\geqslant 1)$ and $t\in I$. Hence, by (8),

$$\begin{aligned}
|\sigma_{n,m} a(x, y)| &= \left|\int_I\int_J a(t, u)\,K_n(x-t)\,K_m(y-u)\,dt\,du\right| \\
&\leqslant \int_I\left|\int_J a(t, u)\,K_m(y-u)\,du\right| K_n(x-t)\,dt \\
&\leqslant C\int_I\left|\int_J a(t, u)\,K_m(y-u)\,du\right|\frac{1}{(n+1)(x-t)^2}\,dt \\
&\leqslant \frac{C 2^{2K}}{(n+1)i^2}\int_I\left|\int_J a(t, u)\,K_m(y-u)\,du\right| dt.
\end{aligned}\tag{15}$$

By Hölder's inequality,

$$\int_J \sup_{n \geqslant r_i,\, m \in \mathbf{N}} |\sigma_{n,m} a(x, y)|^p \, dy$$

$$\leqslant \frac{C_p 2^{2Kp}}{(r_i+1)^p \, i^{2p}} |J|^{1-p} \left(\int_I \int_J \sup_{m \in \mathbf{N}} \left| \int_J a(t, u) \, K_m(y-u) \, du \right| dy \, dt \right)^p. \tag{16}$$

Using again Hölder's inequality and (12) for one dimension and for a fixed t, we obtain

$$\int_J \sup_{m \in \mathbf{N}} \left| \int_J a(t, u) \, K_m(y-u) \, du \right| dy$$

$$\leqslant |J|^{1/2} \left(\int_{\mathbf{T}} \sup_{m \in \mathbf{N}} \left| \int_J a(t, u) \, K_m(y-u) \, du \right|^2 dy \right)^{1/2}$$

$$\leqslant C \, |J|^{1/2} \left(\int_J |a(t, y)|^2 \, dy \right)^{1/2}.$$

By the definition of the rectangle p-atom we conclude that

$$\int_I \int_J \sup_{m \in \mathbf{N}} \left| \int_J a(t, u) \, K_m(y-u) \, du \right| dy \, dt$$

$$\leqslant C \, |J|^{1/2} \, |I|^{1/2} \left(\int_{\mathbf{T}} \int_{\mathbf{T}} |a(t, y)|^2 \, dy \, dt \right)^{1/2}$$

$$\leqslant C_p 2^{-K+K/p-L+L/p}. \tag{17}$$

Using the value of r_i we can establish that

$$(A) \leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} 2^{-K} \frac{2^{2Kp}}{(r_i+1)^p \, i^{2p}} 2^{-L+Lp} 2^{-Kp+K-Lp+L}$$

$$\leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} \frac{1}{i^{2p-\alpha p}} \leqslant C_{p,\alpha} 2^{-r(2p-\alpha p-1)} \tag{18}$$

provided that

$$\alpha < \frac{2p-1}{p} \ (\leqslant 1). \tag{19}$$

Now let us consider (B). It is known that

$$\sigma_{n,m} a(x, y) = \int_I \left(\int_J a(t, u)\, K_m(y-u)\, du \right) K_n(x-t)\, dt$$
$$= \frac{1}{2\pi} \sum_{|k|=0}^{n} \left(1 - \frac{|k|}{n+1}\right) \int_I \left(\int_J a(t, u)\, K_m(y-u)\, du \right) e^{\imath kt}\, dt\, e^{\imath kx}.$$

By the definition of the atom,

$$\left| \int_I \left(\int_J a(t, u)\, K_m(y-u)\, du \right) e^{\imath kt}\, dt \right|$$
$$\leqslant \left| \int_I \left(\int_J a(t, u)\, K_m(y-u)\, du \right) (e^{\imath kt} - 1)\, dt \right|$$
$$\leqslant \int_I \left| \int_J a(t, u)\, K_m(y-u)\, du\, |k|\, |t| \right| dt$$
$$\leqslant |I|\, |k| \int_I \left| \int_J a(t, u)\, K_m(y-u)\, du \right| dt.$$

Therefore

$$\sup_{n < r_i,\, m \in \mathbf{N}} |\sigma_{n,m} a(x, y)|$$
$$\leqslant C \sup_{n < r_i,\, m \in \mathbf{N}} \sum_{|k|=0}^{n} \frac{n+1-|k|}{n+1} |I|\, |k| \int_I \left| \int_J a(t, u)\, K_m(y-u)\, du \right| dt$$
$$\leqslant C_p \sum_{k=0}^{r_i} (r_i - k)\, 2^{-K} \int_I \sup_{m \in \mathbf{N}} \left| \int_J a(t, u)\, K_m(y-u)\, du \right| dt$$
$$\leqslant C_p r_i^2 2^{-K} \int_I \sup_{m \in \mathbf{N}} \left| \int_J a(t, u)\, K_m(y-u)\, du \right| dt. \tag{20}$$

It follows similarly to (16) and (17) that

$$(B) \leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} 2^{-K} r_i^{2p} 2^{-Kp}\, |J|^{1-p}$$
$$\times \left(\int_I \int_J \sup_{m \in \mathbf{N}} \left| \int_J a(t, u)\, K_m(y-u)\, du \right| dy\, dt \right)^p$$
$$\leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} 2^{-K} 2^{2Kp} i^{-2\alpha p} 2^{-Kp} 2^{-L+Lp} 2^{-Kp+K-Lp+L}$$
$$\leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} i^{-2\alpha p} \leqslant C_{p,\alpha} 2^{-r(2\alpha p-1)} \tag{21}$$

whenever

$$\alpha > \frac{1}{2p}. \tag{22}$$

The number α satisfies (19) and (22) if and only if $3/4 < p \leqslant 1$.

Combining (18) and (21) we can establish that, for $3/4 < p \leqslant 1$,

$$\int_{\mathbf{T}\setminus 2^r I} \int_J |\sigma^* a(x, y)|^p \, dx \, dy \leqslant C_p 2^{-\delta r} \tag{23}$$

where C_p depends only on p.

Step 2. *Integrating over* $(\mathbf{T}\setminus 2^r I) \times (\mathbf{T}\setminus J)$. Similarly to (13),

$$\begin{aligned}
&\int_{\mathbf{T}\setminus 2^r I} \int_{\mathbf{T}\setminus J} |\sigma^* a(x, y)|^p \, dx \, dy \\
&\quad \leqslant \sum_{|i| = 2^{r-2}}^{2^K - 1} \sum_{|j| = 1}^{2^L - 1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-L}}^{\pi(j+1)2^{-L}} \sup_{n \geqslant r_i,\, m \geqslant s_j} |\sigma_{n,m} a(x, y)|^p \, dx \, dy \\
&\qquad + \sum_{|i| = 2^{r-2}}^{2^K - 1} \sum_{|j| = 1}^{2^L - 1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-L}}^{\pi(j+1)2^{-L}} \sup_{n < r_i,\, m \geqslant s_j} |\sigma_{n,m} a(x, y)|^p \, dx \, dy \\
&\qquad + \sum_{|i| = 2^{r-2}}^{2^K - 1} \sum_{|j| = 1}^{2^L - 1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-L}}^{\pi(j+1)2^{-L}} \sup_{n \geqslant r_i,\, m < s_j} |\sigma_{n,m} a(x, y)|^p \, dx \, dy \\
&\qquad + \sum_{|i| = 2^{r-2}}^{2^K - 1} \sum_{|j| = 1}^{2^L - 1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-L}}^{\pi(j+1)2^{-L}} \sup_{n < r_i,\, m < s_j} |\sigma_{n,m} a(x, y)|^p \, dx \, dy \\
&\quad = (C) + (D) + (E) + (F)
\end{aligned}$$

where $r_i := [2^K / i^\alpha]$ and $s_j := [2^L / j^\alpha]$ $(i, j \in \mathbf{N})$ with $\alpha > 0$ chosen later.

Similarly to (15) and (17),

$$\begin{aligned}
|\sigma_{n,m} a(x, y)| &= \left| \int_I \int_J a(t, u) \, K_n(x - t) \, K_m(y - u) \, dt \, du \right| \\
&\leqslant C \int_I \int_J |a(t, u)| \, \frac{1}{(n+1)(x-t)^2} \, \frac{1}{(m+1)(y-u)^2} \, dt \, du \\
&\leqslant \frac{C 2^{2K} 2^{2L}}{(n+1)(m+1) \, i^2 j^2} \, |I|^{1/2} \, |J|^{1/2} \left(\int_I \int_J |a(t, u)|^2 \, dt \, du \right)^{1/2} \\
&\leqslant \frac{C_p 2^{K + L + K/p + L/p}}{(n+1)(m+1) \, i^2 j^2}.
\end{aligned}$$

Consequently,

$$(C) \leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} \sum_{j=1}^{2^L-1} 2^{-K-L} \frac{2^{Kp+Lp+K+L}}{(r_i+1)^p (s_j+1)^p i^{2p} j^{2p}}$$
$$\leqslant C_p \sum_{|i|=2^{r-2}}^{2^K-1} \sum_{|j|=1}^{2^L-1} \frac{1}{i^{2p-\alpha p} j^{2p-\alpha p}} \leqslant C_{p,\alpha} 2^{-r(2p-\alpha p-1)} \tag{24}$$

whenever (19) holds.

We get from (14), (17) and (20) that

$$\sup_{n<r_i,\, m\geqslant s_j} |\sigma_{n,m} a(x,y)| \leqslant C_p r_i^2 2^{-K} \sup_{m \geqslant s_j} \int_I \int_J |a(t,u)| \frac{1}{(m+1)(y-u)^2}\, dt\, du$$
$$\leqslant C_p r_i^2 2^{-K} \frac{1}{(s_j+1)} \frac{2^{2L}}{j^2} 2^{-K-L+K/p+L/p}$$
$$\leqslant C_p \frac{2^{K/p+L/p}}{i^{2\alpha} j^{2-\alpha}}.$$

Hence

$$(D) \leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} \sum_{j=1}^{2^L-1} 2^{-K-L} \frac{2^{K+L}}{i^{2\alpha p} j^{(2-\alpha)p}} \leqslant C_{p,\alpha} 2^{-r(2\alpha p-1)} \tag{25}$$

if (19) and (22) are satisfied.

Similarly,

$$\sup_{n\geqslant r_i,\, m<s_j} |\sigma_{n,m} a(x,y)| \leqslant C_p \frac{2^{K/p+L/p}}{j^{2\alpha} i^{2-\alpha}}$$

and

$$(E) \leqslant C_p \sum_{i=2^{r-2}}^{2^K-1} \sum_{j=1}^{2^L-1} 2^{-K-L} \frac{2^{K+L}}{j^{2\alpha p} i^{(2-\alpha)p}} \leqslant C_{p,\alpha} 2^{-r(2p-\alpha p-1)} \tag{26}$$

provided that (19) and (22) are true.

We know that

$$\sigma_{n,m} a(x,y) = \sum_{|k|=0}^{n} \sum_{|l|=0}^{m} \left(1 - \frac{|k|}{n+1}\right)\left(1 - \frac{|l|}{m+1}\right) \hat{a}(k,l) e^{\imath kx + \imath ly}.$$

Next, we establish that

$$|\hat{a}(k, l)| = \left| \frac{1}{(2\pi)^2} \int_I \int_J a(x, y)(e^{-\imath kx} - 1)(e^{-\imath ly} - 1)\, dx\, dy \right|$$

$$\leqslant \frac{1}{(2\pi)^2} \int_I \int_J |a(x, y)|\, |kx|\, |ly|\, dx\, dy$$

$$\leqslant \frac{1}{(2\pi)^2}\, |k|\, |l|\, |I|\, |J|\, |I|^{1/2}\, |J|^{1/2} \left(\int_I \int_J |a(x, y)|^2\, dx\, dy \right)^{1/2}$$

$$\leqslant C_p\, |k|\, |l|\, 2^{-2K - 2L + K/p + L/p}.$$

So

$$\sup_{n < r_i,\, m < s_j} |\sigma_{n, m} a(x, y)| \leqslant C \sup_{n < r_i,\, m < s_j} \sum_{|k| = 0}^{n} \sum_{|l| = 0}^{m} \frac{n + 1 - |k|}{n + 1} \times \frac{m + 1 - |l|}{m + 1}\, |\hat{a}(k, l)|$$

$$\leqslant C_p \sum_{k = 0}^{r_i} \sum_{l = 0}^{s_j} (r_i - k)(s_j - l)\, 2^{-2K - 2L + K/p + L/p}$$

$$\leqslant C_p r_i^2 s_j^2 2^{-2K - 2L + K/p + L/p}.$$

A simple calculation shows that

$$(F) \leqslant C_p \sum_{i = 2^{r-2}}^{2^K - 1} \sum_{j = 1}^{2^L - 1} 2^{-K - L} r_i^{2p} s_j^{2p} 2^{-2Kp - 2Lp + K + L}$$

$$\leqslant C_p \sum_{i = 2^{r-2}}^{2^K - 1} \sum_{j = 1}^{2^L - 1} \frac{1}{i^{2\alpha p} j^{2\alpha p}} \leqslant C_{p, \alpha} 2^{-r(2\alpha p - 1)}$$

supposed that (19) holds.

Combining (24)–(27) we can see that, for $3/4 < p \leqslant 1$,

$$\int_{\mathbf{T} \backslash 2^r I} \int_{\mathbf{T} \backslash J} |\sigma^* a(x, y)|^p\, dx\, dy \leqslant C_p 2^{-\delta r} \tag{28}$$

where C_p depends only on p.

Steps 3 and 4, the integration over $I \times (\mathbf{T} \backslash 2^r J)$ and over $(\mathbf{T} \backslash I) \times (\mathbf{T} \backslash 2^r J)$, are analogous to Steps 1 and 2.

Taking into account (23) and (28), we have proved (7) and also the theorem. ▮

Note that Theorem 3 was proved by the author for the one-parameter case, for the restricted maximal operator and $H_p(\mathbf{T}^2)$ and, moreover, for the two-parameter Walsh–Fourier series (see [20], [19], [21]).

We suspect that Theorem 3 for $p \leqslant 3/4$ is not true though we could not find any counterexample.

It is easy to show that the two-dimensional trigonometric polynomials are dense in $H_1^\sharp$. Hence (11) and the usual density argument (see Marcinkiewicz, Zygmund [14]) imply

COROLLARY 1. *If* $f \in H_1^\sharp$ *then*

$$\sigma_{n,m} f \to f \qquad \text{a.e. as} \quad \min(n, m) \to \infty.$$

Note that this corollary is proved in [20] with another method.

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